Lecture 10. ***Creation of function of Lyapunov for linear systems***

Let linear dynamical system of the following form be given:

$$\dot{x} = Ax + Bu, \; u(t) \equiv 0.$$

We will consider a free system of the form:

$$\dot{x} = Ax. \qquad (6a)$$

Let us choose a function of the following form:

$$V(x) = x^T P x, \qquad (6.7)$$

where $P(n \times n)$ is a constant matrix; $P^T = P$ is a symmetric matrix, i.e. $P_{ij} = P_{ji}$, $\forall i, j = \overline{1, n}$.

Square-law form (type 6.7) will be considered as Lyapunov's function, but firstly it is reasonable to investigate the properties of a square-law form.

10.1. The properties of a square-law form

Sylvester's criteria

Criterion 1. A square-law form is one of fixed positive-sign, if all main diagonal monitors of matrix *P* are positive.

Matrix $P > 0$ is a positive defined matrix *P*, i.e.

$$P_{11} > 0; \; \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} > 0; \ldots \det P > 0.$$

Criterion 2. A square-law form is one of fixed negative-sign, if signs of main diagonal monitors alternate, beginning with negative. $P < 0$ means, that matrix *P* is defined as negative, i.e.

$$P_{11} < 0; \; \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} > 0; \; \begin{vmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{vmatrix} < 0 \; \textit{and so on},$$

it means that signs alternate, beginning with a negative one.

10.2. Creation of Lyapunov's function for linear systems

Let us consider along with system (6.a) the system transported to it:

$$(\dot{x})^T = x^T A^T.$$

Matrix P is chosen as one of fixed positive sign, i.e. $P>0$. The differential of Lyapunov's function is taken as the following (6.7):

$$\frac{dV}{dt} = \dot{x}^T Px + x^T P\dot{x} = x^T A^T Px + x^T PAx = x^T (A^T P + PA)x = -x^T Qx,$$

where $Q(n \times n)$ is a symmetric matrix, it means that $Q = Q^T$, besides $Q>0$ is a matrix of fixed positive sign. Hence,

$$A^T P + PA = -Q. \tag{6.8}$$

So, equation (6.8) is Lyapunov's matrix equation. Solution of matrix equation is matrix P.

THEOREM. To do matrix A stable, it is necessary and sufficient, for matrix equation (6.8) to have positive-defined solution $P > 0$ at any positive-defined matrix $Q > 0$.

Matrix equation (6.8) is reduced to the system $\frac{(n+1)n}{2}$ of linear algebraic equations, where "*n*" is an order of the system.

Example 6.9. Let a linear dynamical system of the 2*nd* order of the following type be given:

$$\begin{cases} \dot{x}_1 = -2x_1 \\ \dot{x}_2 = -5x_2 \end{cases}$$

where $x \in R^2$.

The following type of Lyapunov's function of $V(x)$ is given:

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

or

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) = \frac{1}{2}x^T Px, where\, P = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad x = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}.$$

You should define stability of system by Lyapunov.

Algorithm and solution

1. We will be convinced that the given function *V(x)* is one of fixed positive-sign, i.e.

$$\begin{cases} V(x) > 0 \; if \;\; x \neq 0 \\ V(x) = 0 \; if \;\; x = 0 \end{cases}.$$

2. It is necessary to define *the sign of its full derivative*.

$$\frac{dV}{dt} \overset{\Delta}{=} (\nabla V)^T \frac{dx}{xt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt};$$

$$\frac{dV}{dt} = x_1\dot{x}_1 + x_2\dot{x}_2 = -2x_1^2 - 5x_2^2 = -(2x_1^2 + 5x_2^2).$$

$$\frac{dV}{dt} = -(2x_1^2 + 5x_2^2) < 0.$$

Conclusion: hence, the system is *asymptotically stable* at any value of x_1, x_2.

Example 6.10. Let the following linear dynamical system of the 2nd order be given:

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = -x_2 \end{cases},$$

where $x \in R^2$.

The following Lyapunov's function is given:

$$\frac{dV}{dt} = \frac{1}{2}(x_1^2 + x_2^2) > 0.$$

You should define stability of system by Lyapunov.

Solution:

$$\frac{dV}{dt} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^2 - x_2^2 < 0 \; in\, case \;\; x_2 > x_1.$$

Hence, the system is asymptotically stable according to Lyapunov at condition when $x_2 > x_1$.

This method was generalized by American scientist Richard Kalman.

THEOREM of Kalman. The real parts of characteristic roots of matrix *A* will be less than zero $\alpha < 0$ only in one case: when for any positively defined symmetric matrix *Q*, there exists positively defined symmetric matrix *P*, which is the only solution of the following type equation:

$$-2\,\alpha P + A^T P + PA = -Q,$$

where $Re\,\lambda_i(A) \le \alpha_i$; $\alpha = const.$

Example 6.11. Let the following linear dynamical system of the 2nd order is given:

$$\begin{cases} \dot{x} = 5x_1 - 3x_2 \\ \dot{x} = -4x_1 + 2x_2 \end{cases}.$$

The following Lyapunov's function is given:

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2).$$

You should define system stability according to Lyapunov.

Algorithm and solution

1. We will be convinced that the given function *V(x)* is one of fixed positive-sign, i.e.

$$\begin{cases} V(x) > 0 \; if \;\; x \ne 0 \\ V(x) = 0 \; if \;\; x = 0 \end{cases}.$$

2. It is necessary to define *the sign of its full derivative*.

$$\frac{dV}{dt} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(5x_1 - 3x_2) + x_2(-4x_1 + 2x_2) < 0$$

$$5x_1^2 - 3x_1x_2 - 4x_1x_2 + 2x_2^2 < 0;$$

$$5x_1^2 \ne 0 \Rightarrow \frac{5x_1^2}{5x_1^2} - \frac{7x_1x_2}{5x_1^2} + \frac{2x_2^2}{5x_1^2}$$

3. Let us introduce the designation $\frac{x_2}{x_1} = z$ and obtain:

$$\frac{2}{5}z^2 - \frac{7}{5}z \quad +1 < 0;$$

$$z^2 - \frac{7 \times 5}{2 \times 5}z + \frac{5}{2} < 0;$$

$$z^2 - \frac{7}{2}z \quad + \frac{2}{5}z + \frac{5}{2} < 0;$$

$$z_{1,2} = \frac{7}{4} \pm \sqrt{\frac{49}{16} - \frac{5}{2}}$$

$$z_{1,2} = \frac{7}{4} \pm \frac{3}{4}; \; z_1 = 1, \; z_2 = \frac{10}{4}$$

Conclusion: this system is *asymptotically stable according to Lyapunov* in case when $x_1 \in (0.4x_2, x_2)$ (fig.).

Geometrical interpretation:

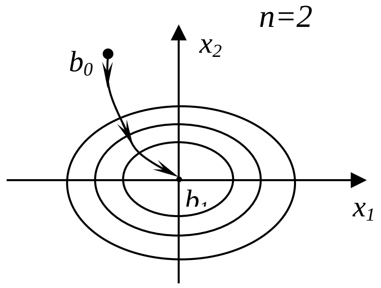

Fig. – The movement of a system asymptotically is steady across Lyapunov